Graph of Linear Transformations over $\mathbb{R}$

Ayman Badawi and Yasmine El-Ashi

Department of Mathematics & Statistics, The American University of Sharjah, P.O. Box 26666, Sharjah, United Arab Emirates,
abadawi@aus.edu, g00007313@alumni.aus.edu

Abstract. Let $m, n \geq 1$ be positive integers, X and Y be finite dimensional vector spaces over $\mathbb{R}$ (the set of all real numbers), where $dim_{\mathbb{R}}(X) = m$ and $dim_{\mathbb{R}}(Y) = n$. In this paper, we introduce a new graph, denoted by $G_{m,n}$, with vertex set $V = \{T : X \to Y \mid T$ is a nontrivial linear transformation$\}$.

Keywords: zero-divisor graph, total graph, unitary graph, dot product graph, annihilator graph, linear transformations graph

1 Introduction

Throughout this paper, R denotes a commutative ring with $1 \neq 0$ and $Z(R)$ denotes the set of all zero-divisors of R. Let $a \in Z(R)$ and let $ann_R(a) = \{r \in R \mid ra = 0\}$. In 2014, A. Badawi [26] introduced the annihilator graph of R. We recall from [26] that the annihilator graph of R is the (undirected) graph $AG(R)$ with vertices $Z(R)^* = Z(R) \setminus \{0\}$, and two distinct vertices x and y are adjacent if and only if $ann_R(xy) \neq ann_R(x) \cup ann_R(y)$. See the survey article [23]. It follows that each edge (path) of the classical zero-divisor of R is an edge (path) of $AG(R)$. For further investigations of $AG(R)$, see [19], [50], and [56]. We remind the reader that the *zero-divisor graph* of R as in [17] is the (simple) graph $\Gamma(R)$ with vertices $Z(R) \setminus \{0\}$, and distinct vertices x and y are adjacent if and only if $xy = 0$. This concept is due to Beck [28], who let all the elements of R be vertices. The zero-divisor graph of a ring R has been studied extensively by many authors, for example see([2]-[9], [12], [21]-[22], [37]-[43], [46]-[53], [57]). David. F. Anderson and the first-named author [13] introduced the *total graph* of R, denoted by $T(\Gamma(R))$. We recall from [13] that the total graph of a commutative ring R is the (simple) graph $\Gamma(R)$ with vertices R, and distinct vertices x and y are adjacent if and only if $x + y \in Z(R)$. The total graph (as in [13]) has been investigated in [8], [7], [6], [5], [45], [47], [51], [34] and [55]; and several variants of the total graph have been studied in [4], [14], [15], [16], [20], [27], [33], [30], [31], [32], [35], [36], and [44]. In 2015, A. Badawi, investigated the *total dot product graph* of R [25]. In this case $R = A \times A \times \cdots \times A$ (n times), where A is a commutative ring with nonzero identity, and $1 \leq n < \infty$ is an integer. The *total dot product graph* of R is the (undirected) graph denoted by $TD(R)$, with vertices $R^* = R \setminus \{(0, 0, \ldots 0)\}$. Two distinct vertices are adjacent if and only if $x \cdot y = 0 \in A$, where $x \cdot y$ denote the normal dot product of x

and y. The *zero-divisor dot product graph* of R is the induced subgraph $ZD(R)$ of $TD(R)$ with vertices $Z(R)^* = Z(R) \setminus \{(0,0,...,0)\}$. It follows that each edge (path) of the classical zero-divisor graph $\Gamma(R)$ is an edge (path) of $ZD(R)$. In [25], both graphs $TD(R)$ and $ZD(R)$ are studied. The total dot product graph was recently further investigated in [1].

There has been considerable attention in the literature to graphs from rings and groups; see the survey articles [11], [10], [38] and [45]. For other types of graphs attached to groups and rings, for example see [6], [8],[27], [37], [39]–[43], and [44].

In this paper, we introduce a connection between graph theory and linear transformations of finite dimensional vector spaces over $\mathbb{R}$ (the ring of all real numbers). Since every finite dimensional vector space over $\mathbb{R}$ with dimension h is isomorphic to $\mathbb{R}^h$, let $m, n \geq 1$ be positive integers and $L = \{t : \mathbb{R}^m \to \mathbb{R}^n \mid t$ is a nontrivial linear transformation from $\mathbb{R}^m$ into $\mathbb{R}^n\}$. If $g, v \in L$, then we say that g is equivalent to v, and we write $g \sim v$ if and only if $Ker(g) = Ker(v)$. Clearly, $\sim$ is an equivalence relation on L. For each $v \in L$, the set $[v] = \{w \in L | w \sim v\}$ is called the *equivalence class* of v. Let $V_{m,n}$ be the set of all equivalence classes of $\sim$. For positive integers $m, n \geq 1$, let $G_{m,n}$ be a simple undirected graph with vertex set $V_{m,n}$ such that two distinct vertices $[h], [w] \in V_{m,n}$ are adjacent if and only if $Ker(h) \cap Ker(w) \neq \{(0, \cdots, 0)\} \subset \mathbb{R}^m$.

We recall the following definitions.

Definition 1. *Let G be a graph.*

1. *Two vertices v_1, v_2 of G are said to be adjacent in G if v_1, v_2 are connected by an edge of G and we write $v_1 - v_2$. For vertices x and y of G.*
2. *We define $d(x, y)$ to be the length of a shortest path from x to y ($d(x, x) = 0$ and $d(x, y) = \infty$ if there is no path).*
3. *The diameter of G is $diam(G) = sup\{\ d(x, y) \mid x$ and y are vertices of $G\}$.*
4. *The girth of G, denoted by $gr(G)$, is the length of a shortest cycle in G ($gr(G) = \infty$ if G contains no cycles).*
5. *G is connected if there is a path in G from u to v for every $u, v \in V$.*
6. *G is disconnected, if there exist at least two vertices $u, v \in V$ that are not joined by a path.*
7. *G is totally disconnected if no two vertices of G are adjacent.*

Recall that a graph G is called complete if every two vertices of G are adjacent. We denote the complete graph on n vertices by K_n,

2 Results

Remark 1. If a graph G has one vertex, then we say that G is totally disconnected. Note that some authors state that such graph is connected.

We have the following result.

Theorem 1. *The undirected graph $G_{m,1}$ is totally disconnected if and only if $m = 1$ or $m = 2$. Furthermore, if $m = 1$, then $V_{1,1} = \{[t]\}$ for some $t \in L$.*

Proof. Assume $m = 1$. Let $[t] \in V_{1,1}$. Since $t \in L$ (i.e., t is a nontrivial linear transformation from $\mathbb{R}$ into $\mathbb{R}$), we conclude that $dim(Range(t)) = 1$. Since $dim(Ker(t)) + dim(Range(t)) = m = 1$ and $dim(Range(t)) = 1$, we conclude that $Ker(t) = \{0\}$. Thus $f \in [t]$ for every $f \in L$. Hence $V_{1,1} = \{[t]\}$ for some $t \in L$. Thus $G_{1,1}$ is totally disconnected by Remark 1.

Assume $m = 2$. Let $[t], [f] \in V_{2,1}$ be two distinct vertices. Since $t, f \in L$ (i.e., t, f are nontrivial linear transformations from $\mathbb{R}^2$ into $\mathbb{R}$), we conclude that $dim(Range(t)) = dim(Range(t)) = 1$. Since $dim(Ker(t)) + dim(Range(t)) = m = 2$ and $dim(Range(t)) = 1$, we conclude that $dim(Ker(t)) = 1$. Similarly, $dim(Ker(f)) = 1$. Since $t, f \in L$, and $dim(Ker(t)) = dim(Ker(f)) = 1$, we conclude that $Ker(t)$ and $Ker(f)$ are distinct lines passing through the origin $(0, 0)$. Thus $Ker(t) \cap Ker(f) = \{(0, 0)\}$. Hence $[t], [f]$ are nonadjacent. Thus $G_{2,1}$ is totally disconnected.

Now assume $m > 2$. We show that $G_{m,1}$ is connected. Let, $[t], [w] \in V_{m,1}$ be two distinct vertices. We show that $\ker(f) \cap \ker(k) \neq \{(0, \cdots, 0)\}$ for some $f \in [t]$ and $k \in [w]$. Let $\mathbf{M}_f$ be the standard $1 \times m$ matrix representation of f for some $f \in [t] \in V_{m,1}$ and $\mathbf{M}_k$ be the standard $1 \times m$ matrix representation of k for some $k \in [w] \in V_{m,1}$. By hypothesis, $\mathbf{M}_f$ is not *row-equivalent* to $\mathbf{M}_k$. Say, $\mathbf{M}_f = \begin{bmatrix} f_{11} \; f_{12} \; \cdots \; f_{1m} \end{bmatrix}$ and $\mathbf{M}_k = \begin{bmatrix} k_{11} \; k_{12} \; \cdots \; k_{1m} \end{bmatrix}$

Let, $\mathbf{M}_{fk} = \begin{bmatrix} \mathbf{M}_f \\ \mathbf{M}_k \end{bmatrix}$ and consider the system, $\mathbf{M}_{fk}\mathbf{x} = \mathbf{0}$, that is,

$$\begin{bmatrix} f_{11} \; f_{12} \cdots f_{1m} \\ k_{11} \; k_{12} \cdots k_{1m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Since, $m > 2$, the number of equations $<$ the number of unknown variables. Hence, the system $\mathbf{M}_{fk}\mathbf{x} = \mathbf{0}$ has infinitely many solutions. Therefore, $\ker(f) \cap \ker(k) \neq \mathbf{0}$, that is, the vertices $[t]$ and $[w]$ are adjacent. Further, since $[t]$, $[w]$ were chosen randomly, we conclude that the graph $G_{m,1}$ is complete for $m > 2$.

Theorem 2. *For $m = 1$ or $m = 2$, the undirected graph $G_{2,n}$ is totally disconnected for every positive integer $n \geq 1$.*

Proof. Assume $m = 1$ and $n \geq 1$ be a positive integer. Then by the proof of Theorem 1, we conclude that $V_{1,n} = \{[t]\}$ for some $t \in L$. Hence $V_{1,n}$ is totally disconnected by Remark 1.

Assume $m = 2$, and let $[t], [w] \in V$ be two distinct vertices. We want to show $\ker(f) \cap \ker(k) = 0$ for some $f \in [t]$ and $k \in [w]$. We may assume that neither $Ker(f) = 0$ nor $Ker(k) = 0$. Hence $dim(Ker(f)) = dim(Ker(k)) = 1$. Thus $Ker(f) \cap Ker(k) = \{(0, 0)\}$. Since $[f]$, $[k]$ were chosen randomly, we conclude that the graph $G_{2,n}$ is totally disconnected for $m = 2$.

Theorem 3. *The graph $G_{m,n}$ is complete if and only if $m \geq 2n + 1$.*

Proof. Let $[t], [w] \in V$ such that $Ker(f) \neq 0$ and $Ker(k) \neq 0$ for some $f \in [t]$ and $k \in [w]$. Let $\mathbf{M}_f$ be the standard $n \times m$ matrix representation of $[f]$, $\mathbf{M}_k$ be the standard $n \times m$ matrix representation of $[k]$, and let $\mathbf{M}_{fk} = \begin{bmatrix} \mathbf{M}_f \\ \mathbf{M}_k \end{bmatrix}$

Assume, $(x_1, x_2, \cdots, x_m) \in \mathbf{R}^m$ is a solution to $\mathbf{M}_{fk}\mathbf{x} = 0$, that is,

$$\begin{bmatrix} \mathbf{M}_f \\ \mathbf{M}_k \end{bmatrix}_{2n \times m} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}_{m \times 1} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{2n \times 1}$$

Let $r = \text{rank}(\mathbf{M}_{fk})$.

Assume $m \geq 2n + 1$. We show $\ker(f) \cap \ker(k) \neq 0$. Since $r \leq 2n$ and $m \geq 2n + 1$, we have number of equations $<$ number of unknown variables. Hence, the system $\mathbf{M}_{fk}\mathbf{x} = 0$ has infinitely many solutions, or $\text{null}(\mathbf{M}_{fk}) \neq 0$. Therefore, $\ker(f) \cap \ker(k) \neq 0$, that is the vertices $[t]$ and $[w]$ are adjacent. Since $[t]$ and $[w]$ are chosen randomly, we conclude that the graph $G_{m,n}$ is complete for $m \geq 2n + 1$.

Conversaly, assume that $G_{m,n}$ is complete. We show that $m \geq 2n+1$. Suppose that $m < 2n + 1$. We show that $G_{m,n}$ is not complete. Let $[t], [w] \in V$ such that $Ker(f) \neq 0$ and $Ker(k) \neq 0$ for some $f \in [t]$ and $k \in [w]$.

Case I: Suppose $r = m$.
We conclude that $\mathbf{M}_{fk}$ has m independent rows, say $R_1, R_2, \cdots, R_m$.
Consider the system,

$$\begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Since $\begin{bmatrix} R_1 & R_2 & \cdots & R_m \end{bmatrix}^T$ is an invertible $m \times m$ matrix, we have $\text{null}\big(\begin{bmatrix} R_1 & R_2 & \cdots & R_m \end{bmatrix}\big)^T = (0, 0, \cdots, 0)$. Thus $\ker(t) \cap \ker(w) = 0$. Hence the vertices $[t]$ and $[w]$ are nonadjacent

Case II: Suppose $r < m$. Thus we have the following system:

$$\begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_r \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Since number of equations $<$ number of unknown variables, we conclude that $\text{null}\left(\begin{bmatrix} R_1 & R_2 & \cdots & R_r \end{bmatrix}^T\right) \neq (0, 0, \cdots, 0)$. This implies $\ker(f) \cap \ker(k) \neq 0$. Hence the vertices $[t]$ and $[w]$ are adjacent.

Since the vertices $[t]$ and $[w]$ can either be adjacent or nonadjacent, we conclude that the graph $G_{m,n}$ is not complete for every $1 \leq m < 2n+1$.

Theorem 4. *Consider the undirected graph $G_{m,n}$. Assume $m \leq n$ and $m \neq 1$ or $m \neq 2$. Then $G_{m,n}$ is connected and $diam(G_{m,n}) = 2$.*

Proof. Let $[t]$, $[w] \in V$ such that $[t]$ and $[w]$ are nonadjacent. Choose $f \in [t]$ and $k \in [w]$. Then $\text{rank}\,(M_f) \neq m$ and $\text{rank}\,(M_k) \neq m$, where M_f and M_k are the standard matrix representations of f and k, with size $n \times m$.
Assume $\text{rank}\,(M_f) = m - i$, where $i \in \mathbf{N}$, $i \neq 1$, and $\text{rank}\,(M_k) = m - j$, where $j \in \mathbf{N}$, $j \neq 1$. Then choose any non-zero row from M_f or M_k, say Y, to form the $n \times m$ matrix M_d, where:

$$M_d = \begin{bmatrix} Y \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

is the standard matrix representation of some $d \in [h] \in V_{m,n}$, such that $[t] - [h] - [w]$.

Assume that $\text{rank}\,(M_f) = m - 1$ and $\text{rank}\,(M_k) = m - 1$. Then M_f has $m-1$ independent rows, $R_1, R_2, \ldots, R_{m-1}$. Since $[t]$ and $[w]$ are nonadjacent, M_k has one row say R such that, $\{R_1, R_2, \ldots, R_{m-1}, R\}$ is an independent set which forms a basis for $\mathbf{R}^m$. Let $K \neq R$ be a non-zero row in M_k. Hence $K \in \text{rowspace}\,(M_k)$. Since $K \in \mathbf{R}^m$, we have:

$$K = c_1 R_1 + c_2 R_2 + \cdots + c_{m-1} R_{m-1} + c_m R$$

Let $Y = K - c_m R$. Thus $Y \in \text{rowspace}\,(M_k)$, (since both K and $c_m R$ are $\in \text{rowspace}\,(M_k)$), and $Y \in \text{rowspace}\,(M_f)$. Let $M_d = \begin{bmatrix} Y \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n \times m}$, be the standard matrix representation of some $d \in [h] \in V_{m,n}$. Since $Y \in \text{rowspace}\,(M_f)$, Y becomes a zero row through row operations using the rows in M_f. Thus $\text{null}\,(M_{fd}) \neq 0$, since $\text{rank}\,(M_{fd}) = m - 1$. Hence $\ker\,(f) \cap \ker\,(d) \neq 0$. Hence $[t]$, $[h]$ are connected by an edge. Similarly, since $Y \in \text{rowspace}\,(M_k)$, Y becomes a zero row through row operations using the rows in M_k. Thus $\text{null}\,(M_{kd}) \neq 0$, since $\text{rank}\,(M_{kd}) = m - 1$. Hence $\ker\,(d) \cap \ker\,(k) \neq 0$. Thus $[h]$ and $[w]$ are adjacent. Therefore, we have $[t] - [h] - [w]$.

Example 1. Suppose $m = 3$ and $n = 4$. So we are considering the graph $G\left([t] : \mathbf{R}^3 \to \mathbf{R}^4\right)$, where $m \leq n$, and $m \neq 1$ or $m \neq 2$, as given in Theorem 4. Let $[T], [L] \in V$, such

that $[T]$ and $[L]$ are not adjacent ($\ker(T) \cap \ker(L) = 0_{m=3}$), and $[T] \neq 0, [L] \neq 0$. Let $f \in [T]$, and $k \in [L]$. Since $[T]$ and $[L]$ are non-trivial vertices, then $\operatorname{rank}(M_f) \neq m$ and $\operatorname{rank}(M_k) \neq m$, where M_f and M_k are the standard matrix representations of f and k.
Suppose,

$$M_f = \begin{bmatrix} 1\,0\,0 \\ 0\,1\,1 \\ 0\,0\,0 \\ 0\,0\,0 \end{bmatrix}_{4\times 3}, M_k = \begin{bmatrix} 0\,0\,0 \\ 0\,0\,1 \\ 1\,1\,0 \\ 0\,0\,0 \end{bmatrix}_{4\times 3}$$

Let $M_{fk} = \begin{bmatrix} M_f \\ M_k \end{bmatrix}_{8\times 3}$
It can be easily seen that $\operatorname{rank}(M_{fk}) = 3$, which implies that $\operatorname{null}(M_{fk}) = 0$. Therefore, $\ker(f) \cap \ker(k) = 0$, that is the vertices $[T]$ and $[L]$ are not adjacent. We have:
$\operatorname{rank}(M_f) = 2 = 3 - 1 = m - 1$, and $\operatorname{rank}(M_k) = 2 = 3 - 1 = m - 1$.

Then M_f has 2 independent rows R_1 and R_2, such that $R_1 = \begin{bmatrix} 1\,0\,0 \end{bmatrix}$ and $R_2 = \begin{bmatrix} 0\,1\,1 \end{bmatrix}$. The vertices $[T]$ and $[L]$ are not adjacent, thus M_k has one row R, such that $\{R_1, R_2, R\}$ are independent and form a basis for $\mathbf{R}^m$, where $m = 3$. In this example, $R = \begin{bmatrix} 0\,0\,1 \end{bmatrix}$. Let $K \neq R$ be a non-zero row in M_k, $K = \begin{bmatrix} 1\,1\,0 \end{bmatrix}$. $K \in \operatorname{rowspace}(M_k)$ and since $K \in \mathbf{R}^3$ it can be written as a linear combination of $\{R_1, R_2, R\}$ as follows:

$$K = 1.R_1 + 1.R_2 - R = \begin{bmatrix} 1\,0\,0 \end{bmatrix} + \begin{bmatrix} 0\,1\,1 \end{bmatrix} - \begin{bmatrix} 0\,0\,1 \end{bmatrix} = \begin{bmatrix} 1\,1\,0 \end{bmatrix}$$

Let $Y = K - (-1).R = K + R = \begin{bmatrix} 1\,1\,0 \end{bmatrix} + \begin{bmatrix} 0\,0\,1 \end{bmatrix} = \begin{bmatrix} 1\,1\,1 \end{bmatrix}$.

This implies $Y \in \operatorname{rowspace}(M_k)$ and $Y \in \operatorname{rowspace}(M_f)$. Let $M_d = \begin{bmatrix} Y \\ 0 \\ 0 \\ 0 \end{bmatrix}_{4\times 3} = \begin{bmatrix} 1\,1\,1 \\ 0\,0\,0 \\ 0\,0\,0 \\ 0\,0\,0 \end{bmatrix}_{4\times 3}$, be the standard matrix representation of some $d \in [W]$.
Since $Y \in \operatorname{rowspace}(M_f)$, Y becomes a zero row through row operations using the rows in M_f. Thus $\operatorname{null}(M_{fd}) \neq 0$ since $\operatorname{rank}(M_{fd}) = 2$. Hence $\ker(T) \cap \ker(W) \neq 0$. Hence $[T], [W]$ are adjacent. Similarly, since $Y \in \operatorname{rowspace}(M_k)$, Y becomes a zero row through row operations using the rows in M_k. Hence $\operatorname{null}(M_{kd}) \neq 0$ since $\operatorname{rank}(M_{kd}) = 2$. Thus $\ker(L) \cap \ker(W) \neq 0$. Thus $[W], [L]$ are adjacent. Therefore, we have $[T] - [W] - [L]$.

Theorem 5. *Consider the undirected graph $G_{m,n}$. Assume that $n < m \leq 2n$ and $m \neq 1$ or $m \neq 2$. Then $G_{m,n}$ is connected and diam$(G_{m,n}) = 2$.*

Proof. Let $[T], [L] \in V$, such that $[T]$ and $[L]$ are not adjacent ($\ker(T) \cap \ker(L) = 0_m$), and $[T] \neq 0$, $[L] \neq 0$. Let, $f \in [T]$ and $k \in [L]$, then $\operatorname{rank}(M_f) < m$ and

$\operatorname{rank}(M_k) < m$, where M_f and M_k are the standard matrix representations of f and k, with size $n \times m$.

Assume that $n+1 < m \leq 2n$. Then $\operatorname{rank}(M_f) = n - i$, where $i = 0, 1, 2, \ldots$, and $\operatorname{rank}(M_k) = n - j$, where $j = 0, 1, 2, \ldots$. Thus we can choose any non-zero row from M_f or M_k, say Y, to form the $n \times m$ matrix M_d, where:

$$M_d = \begin{bmatrix} Y \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

is the standard matrix representation of some $d \in [W]$, such that $[T] - [W] - [L]$.

Assume that $m = n + 1$. Then we have three cases. **Case I**. Assume that $\operatorname{rank}(M_f) = n = m - 1$, and $\operatorname{rank}(M_k) = n - j$, where $j = 1, 2, \ldots$. Then we can choose any non-zero row, say Y from M_f, (Note that M_f is the matrix with the higher rank), to form the $n \times m$ matrix M_d, where:

$$M_d = \begin{bmatrix} Y \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

is the standard matrix representation of some $d \in [W]$, such that $[T] - [W] - [L]$. **Case II**. Assume that $\operatorname{rank}(M_f) = n - i$, where $i = 1, 2, \ldots$ and $\operatorname{rank}(M_k) = n - j$, where $j = 1, 2, \ldots$. In this case any non-zero row Y can be chosen either from M_f or M_k, to form M_d, where:

$$M_d = \begin{bmatrix} Y \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

. is the standard matrix representation of some $d \in [W]$, such that $[T] - [W] - [L]$. **Case III**. Assume that $\operatorname{rank}(M_f) = n$ and $\operatorname{rank}(M_k) = n$. Then M_f has n independent rows $R_1, R_2, \ldots, R_n$. Since $[T]$ and $[L]$ are not adjacent, M_k has one row say R such that, $\{R_1, R_2, \ldots, R_{m-1}, R\}$ is an independent set which forms a basis for $\mathbf{R}^m = \mathbf{R}^{n+1}$. Let $K \neq R$ be a non-zero row in M_k. Hence $K \in \operatorname{rowspace}(M_k)$. Since $K \in \mathbf{R}^{n+1}$, we have:

$$K = c_1 R_1 + c_2 R_2 + \cdots + c_n R_n + c_{n+1} R$$

Let $Y = K - c_{n+1} R$. Hence $Y \in \operatorname{rowspace}(M_k)$, (since both K, $c_{n+1} R \in \operatorname{rowspace}(M_k)$), and $Y \in \operatorname{rowspace}(M_f)$. Let $M_d = \begin{bmatrix} Y \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n \times m}$, be the standard matrix representation of some $d \in [W]$.

Since $Y \in \text{rowspace}(M_f)$, Y becomes a zero row through row operations using the rows in M_f, $\text{null}(M_{fd}) \neq 0$ since $\text{rank}(M_{fd}) = n$. Hence $\ker(T) \cap \ker(W) \neq 0$. Thus $[T], [W]$ are adjacent. Similarly, since $Y \in \text{rowspace}(M_k)$, Y becomes a zero row through row operations using the rows in M_k. Hence $\text{null}(M_{kd}) \neq 0$ since $\text{rank}(M_{kd}) = n$. Thus $\ker(L) \cap \ker(W) \neq 0$. Thus $[W], [L]$ are adjacent. Therefore, we have $[T] - [W] - [L]$.

Example 2. Suppose $m = 4$ and $n = 3$ and consider the graph $G_{4,3}$. Note that $n < m \leq 2n$, $m \neq 1, 2$ and and $m = n+1$. Thus m, n satisfy the given hypothesis in Theorem 5. Let $[T], [L] \in V$, such that $[T]$ and $[L]$ are not adjacent. Let $f \in [T]$, and $k \in [L]$. Then $\text{rank}(M_f) < m$ and $\text{rank}(M_k) < m$, where M_f and M_k are the standard matrix representations of f and k, with size $n \times m = 3 \times 4$. Suppose,

$$M_f = \begin{bmatrix} 1\,0\,0\,0 \\ 0\,1\,0\,1 \\ 0\,0\,1\,0 \end{bmatrix}_{3\times 4}, M_k = \begin{bmatrix} 1\,0\,0\,0 \\ 0\,1\,0\,0 \\ 0\,0\,0\,1 \end{bmatrix}_{3\times 4}$$

Let $M_{fk} = \begin{bmatrix} M_f \\ M_k \end{bmatrix}_{6\times 4}$. It can be easily seen that $\text{rank}(M_{fk}) = 4$, which implies that $\text{null}(M_{fk}) = 0$. Therefore, $\ker(f) \cap \ker(k) = 0$, that is, the vertices $[T]$ and $[L]$ are not adjacent. Hence $\text{rank}(M_f) = 3 = n$, and $\text{rank}(M_k) = 3 = n$. Then M_f has 3 independent rows R_1, R_2, and R_3, such that $R_1 = \begin{bmatrix} 1\,0\,0\,0 \end{bmatrix}$, $R_2 = \begin{bmatrix} 0\,1\,0\,1 \end{bmatrix}$, and $R_3 = \begin{bmatrix} 0\,0\,1\,0 \end{bmatrix}$. The vertices $[T]$ and $[L]$ are not adjacent, thus M_k has one row, $R = \begin{bmatrix} 0\,0\,0\,1 \end{bmatrix}$, such that $\{R_1, R_2, R_3, R\}$ is an independent set which forms a basis for $\mathbf{R}^4$. Let $K \neq R$ be a non-zero row in M_k, $K = \begin{bmatrix} 0\,1\,0\,0 \end{bmatrix}$. Since $K \in \text{rowspace}(M_k)$ and $K \in \mathbf{R}^4$, it can be written as a linear combination of $\{R_1, R_2, R_3, R\}$ as follows:

$$K = 0.R_1 + 1.R_2 + 0.R_3 + (-1).R = \begin{bmatrix} 0\,1\,0\,1 \end{bmatrix} - \begin{bmatrix} 0\,0\,0\,1 \end{bmatrix} = \begin{bmatrix} 0\,1\,0\,0 \end{bmatrix}$$

Let, $Y = K - (-1).R = K + R = \begin{bmatrix} 0\,1\,0\,0 \end{bmatrix} + \begin{bmatrix} 0\,0\,0\,1 \end{bmatrix} = \begin{bmatrix} 0\,1\,0\,1 \end{bmatrix}$.

This implies $Y \in \text{rowspace}(M_k)$ and $Y \in \text{rowspace}(M_f)$. Let, $M_d = \begin{bmatrix} Y \\ 0 \\ 0 \end{bmatrix}_{3\times 4} = \begin{bmatrix} 0\,1\,0\,1 \\ 0\,0\,0\,0 \\ 0\,0\,0\,0 \end{bmatrix}_{3\times 4}$, be the standard matrix representation of some $d \in [W]$.

Since $Y \in \text{rowspace}(M_f)$, Y becomes a zero row through row operations using the rows in M_f. Thus $\text{null}(M_{fd}) \neq 0$, since $\text{rank}(M_{fd}) = 3$. Hence $\ker(T) \cap \ker(W) \neq 0$. Thus $[T], [W]$ are adjacent. Similarly, since $Y \in \text{rowspace}(M_k)$, Y becomes a zero row through row operations using the rows in M_k. Thus $\text{null}(M_{kd}) \neq 0$ since $\text{rank}(M_{kd}) = 3$. Hence $\ker(L) \cap \ker(W) \neq 0$. Thus $[W], [L]$ are adjacent. Therefore, we have $[T] - [W] - [L]$.

Theorem 6. *Assume that $G_{m,n}$ is connected. Then $gr(G_{m,n})) = 3$.*

Proof. $[T], [L] \in V$, such that $[T]$ and $[L]$ are adjacent, $\ker(T) \cap \ker(L) \neq 0$ and $[T] \neq 0, [L] \neq 0$. Let, $f \in [T]$ and $k \in [L]$, then M_f and M_k are the standard matrix representations of f and k with size $n \times m$. Suppose, that each matrix M_f and M_k, is composed of only one row, R_f and R_k that are independent of each other since f and k are in different equivalence classes $[T]$ and $[L]$. M_f and M_k can be written as follows:

$$M_f = \begin{bmatrix} R_f \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n\times m}, M_k = \begin{bmatrix} R_k \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n\times m}$$

Let $Y = R_f + R_k$. Since Y is a linear combination of two linearly independent rows, then the set $\{Y, R_f, R_k\}$ is also linearly independent.

Let $M_d = \begin{bmatrix} Y \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n\times m}$, be the standard matrix representation of some non-trivial linear transformation d. Since Y is independent of both R_f and R_k, M_d is not row-equivalent to either M_f or M_k, hence d is in a different equivalence class from both f and k, say $d \in [W]$. Since $\ker(T) \cap \ker(L) \neq 0$, we have $\mathrm{null}(M_{fk}) \neq 0$, which implies $\mathrm{null}(M_{fd}) \neq 0$ and $\mathrm{null}(M_{kd}) \neq 0$. Therefore, we have, $[T] - [L] - [W] - [T]$. This forms the shortest possible cycle. Hence $gr(G_{m,n})) = 3$.

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